Modeling And Pricing Event Risk
Part III - Enterprise Value

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In this white paper we will build a valuation equation to calculate enterprise value where annualized cash flow follows a jump-diffusion process where cumulative jump count is equal to zero (no jump) or one (jump). To assist us in this endeavor we will use the following hypothetical problem from Part I...

Our Hypothetical Problem

We are tasked with calculating the enterprise value of a non-public company with one major customer using the following model assumptions...

Table 1: Model Assumptions

Symbol	Description	Value	Notes
C_0	Annualized cash flow at time zero	1,000,000	From Part I
μ	Annualized cash flow growth rate - mean	0.0175	From Part I
σ	Annualized cash flow growth rate volatility	0.2500	From Part II
λ	Hazard rate	0.1250	From Part I
π	Market price of risk	0.3267	From Part II
ω	Jump size	0.6000	From Part I
J	Log of one minus the jump size	-0.9163	From Part II
r_f	Risk-free rate	0.0300	From Part II

We will use our model to answer the following questions:

Question 1: What is enterprise value using a time-dependent discount rate?
Question 2: What is enterprise value using an OLS estimate of the discount factor?

Valuation Equations

We will define the variable V_0 to be enterprise value at time zero. Enterprise value is defined as the discounted value of expected cash flow over the time interval $[0, \infty]$. We defined the variable C_t to be annualized cash flow at some future time t and the variable κ_t to be the time-dependent discount rate. The equation for enterprise value is...

$$V_0 = \int_0^\infty \mathbb{E}\Big[C_t\Big] \operatorname{Exp}\Big\{ - \kappa_t \, t \Big\} \delta t \tag{1}$$

Using the parameters from Table 1 above we defined expected annualized cash flow at some future time t to be the following equation...

$$\mathbb{E}\Big[C_t\Big] = C_0 \Bigg[1 - \omega\bigg(1 - \operatorname{Exp}\Big\{ - \lambda \, t \Big\}\bigg)\Bigg] \operatorname{Exp}\Big\{ \mu \, t + \frac{1}{2} \, \sigma^2 t \Big\} \tag{2}$$

Using the parameters from Table 1 above we defined the function ϕ_t^2 to be the variance of the cash flow growth rate at time t divided by time. The equation for variance is...

$$\phi_t^2 = \sigma^2 + J^2 \Bigg[\operatorname{Exp}\Big\{ - \lambda \, t \Big\} - \operatorname{Exp}\Big\{ - 2 \, \lambda \, t \Big\}\Bigg] \, t^{-1} \tag{3}$$

Using the parameters from Table 1 above we defined the variable κ_t to be the time-dependent discount rate. The equation for discount rate is...

$$\kappa_t = r_f + \pi\, \phi_t \tag{4}$$

Note that the time-dependent discount rate as defined by Equations (3) and (4) above is nonlinear. When the variable κ_t in Equation (1) above is nonlinear then the integral cannot be solved explicitly. To get around this problem we will solve the enterprise value equation via the following approximations...

Table 2: Valuation Equation Solutions

1	Numerical integration
2	Linear regression (a.k.a. ordinary least squares)

Solution: Numerical Integration

We will convert the continuous time integral in Equation (1) above into a discrete time summation. We will define the change in time to be the following discrete time equation...

$$dt = \text{one quarter} = \frac{1}{4} \tag{5}$$

Using Equation (5) above the discrete time approximation of the continuous time valuation Equation (1) above becomes...

$$V_0 = \sum_{n=1}^{m} \frac{1}{4}\, \mathbb{E}\Big[C_{\frac{n}{4}}\Big] \Big\{ - \kappa_{\frac{n}{4}} \times \frac{n}{4} \Big\} \text{ ...where m = \# quarters until the PV of cash flow} \approx 0 \tag{6}$$

Using the valuation equations above the first ten rows of our approximation via numerical integration are...

A	B	C	D	E	F
n	time	C(t)	variance	K(t)	PV
1	0.25	994	0.1626	0.1617	239
2	0.50	987	0.1581	0.1599	228
3	0.75	982	0.1537	0.1581	218
4	1.00	976	0.1496	0.1563	209
5	1.25	971	0.1456	0.1547	200
6	1.50	966	0.1418	0.1530	192
7	1.75	961	0.1382	0.1515	184
8	2.00	956	0.1348	0.1499	177
9	2.25	952	0.1316	0.1485	170
10	2.50	948	0.1284	0.1471	164

Using Row 6 of the table above as an example...

Column	Calculation
A	n = 6
B	$n \div 4 = 1.50$
C	$1000 \times [1 - 0.60(1 - \text{Exp}\{-0.1250 \times 1.50\})] \times \text{Exp}\{0.0175 \times 1.50 + 0.50 \times 0.2500^2 \times 1.50\} = 966$
D	$0.2500^2 + (-0.9163)^2 \times [\text{Exp}\{-0.1250 \times 1.50\} - \text{Exp}\{-2 \times 0.1250 \times 1.50\}] \times 1.5^{-1} = 0.1418$
E	$0.0300 + \sqrt{0.1418} \times 0.3267 = 0.1530$
F	$0.25 \times 966 \times \text{Exp}\{-0.1530 \times 1.50\} = 192$

The Answer to Question 1: If we sum column F in the table above then the enterprise value of our hypothetical company is...

$$V_0 = \$8,770,000 \tag{7}$$

Note that column F is in thousands.

Solution: Ordinary Least Squares

We will define the variable Γ_t to be the discount factor at time t. The discount factor is defined as the present value at time zero of one dollar expected to be received at time t. Using the parameters in Table 1 above the equation for the discount factor is...

$$\Gamma_t = \mathrm{Exp}\left\{ - \kappa_t \, t \right\} \tag{8}$$

Using Equation (8) above we can rewrite valuation Equation (1) above as...

$$V_0 = \int\limits_0^\infty \mathbb{E}\left[C_t\right] \Gamma_t \, \delta t \tag{9}$$

We want to derive a linear equation for the discount factor (via ordinary least squares) so that we can solve valuation Equation (9) above in closed-form. As noted above the discount factor is a nonlinear equation. To improve the goodness of fit (i.e. r-squared) we want to regress on the log of the discount factor rather than the discount factor itself. We will define the variable Y_t to be the log of the discount factor at time t. Using Equation (8) above the equation for the log of the discount factor is...

$$Y_t = \ln\left(\Gamma_t\right) = -\kappa_t \, t \tag{10}$$

We will define the variable $\hat{Y}_t$ to be the ordinary least squares estimate of the log of the discount factor as defined by Equation (10) above. The equation for the ordinary least squares estimate is...

$$\hat{Y}_t = \beta \, t \tag{11}$$

We will define the variable ϵ_t to be the regression error, which is the difference at time t between the actual discount factor and the ordinary least squares estimate of the discount factor. Using Equations (10) and (11) above the equation for the regression error is...

$$\epsilon_t = \hat{Y}_t - Y_t = \beta \, t + \kappa_t \, t \tag{12}$$

We will define the variable SSE to be the sum of squared errors over the time interval $[0, T]$. Using Equation (12) above the equation for the sum of squared errors is...

$$\begin{aligned} SSE &= \sum_{t=0}^{T} \epsilon_t^2 \\ &= \sum_{t=0}^{T} \left(\beta \, t + \kappa_t \, t\right)^2 \\ &= \sum_{t=0}^{T} \left(\beta^2 t^2 + \kappa_t^2 t^2 + 2\beta\kappa_t t^2\right) \\ &= \beta^2 \sum_{t=0}^{T} t^2 + \kappa_t^2 \sum_{t=0}^{T} t^2 + 2\beta \sum_{t=0}^{T} \kappa_t t^2 \end{aligned} \tag{13}$$

Note that the derivative of Equation (13) above with respect to the risk-adjusted cost of capital at time t, which is the variable κ_t, is...

$$\frac{\delta SSE}{\delta \kappa} = 2\beta \sum_0^T t^2 + 2 \sum_0^T \kappa_t t^2 \tag{14}$$

To minimize the sum of squared errors we set Equation (14) above equal to zero and solve for the regression coefficient β. The equation that minimizes the sum of squared errors is...

$$\frac{\delta SSE}{\delta \kappa} = 2\beta \sum_0^T t^2 + 2 \sum_0^T \kappa_t t^2 = 0 \tag{15}$$

The solution to Equation (15) above is...

$$\beta = -\sum_{0}^{T} \kappa_t t^2 \Bigg/ \sum_{0}^{T} t^2 \tag{16}$$

Using the regression coefficient in Equation (15) above we can rewrite valuation Equation (1) above as...

$$V_0 = \int_0^\infty \mathbb{E}\Big[C_t\Big] \,\mathrm{Exp}\Big\{\beta\, t\Big\}\,\delta t \tag{17}$$

Using expected annualized cash flow Equation (2) above we can rewrite Equation (17) above as...

$$\begin{aligned} V_0 &= \int_0^\infty C_0 \left[1 - \omega\left(1 - \mathrm{Exp}\Big\{-\lambda\, t\Big\}\right)\right] \mathrm{Exp}\left\{\mu\, t + \frac{1}{2}\,\sigma^2 t\right\} \mathrm{Exp}\Big\{\beta\, t\Big\}\,\delta t \\ &= C_0 \left[\Big(1 - \omega\Big) \int_0^\infty \mathrm{Exp}\left\{\left(\mu + \beta + \frac{1}{2}\,\sigma^2\right) t\right\} \delta t + \omega \int_0^\infty \mathrm{Exp}\left\{\left(\mu - \lambda + \beta + \frac{1}{2}\,\sigma^2\right) t\right\} \delta t\right] \end{aligned} \tag{18}$$

Using Appendix Equations (22) and (23) below the solution to Equation (18) above is...

$$V_0 = C_0 \left[-\Big(1 - \omega\Big)\left(\mu + \beta + \frac{1}{2}\,\sigma^2\right)^{-1} - \omega\left(\mu - \lambda + \beta + \frac{1}{2}\,\sigma^2\right)^{-1}\right] \text{ ...when... } \mu + \beta + \frac{1}{2}\,\sigma^2 < 0 \tag{19}$$

Using the valuation equations above the first ten rows of our beta calculation (Equation (16)) above are...

A	B	C	D	E	F
n	time	variance	K(t)	num	denom
1	0.00	0.1674	0.1637	0.0000	0.0000
2	0.25	0.1626	0.1617	0.0101	0.0625
3	0.50	0.1581	0.1599	0.0400	0.2500
4	0.75	0.1537	0.1581	0.0889	0.5625
5	1.00	0.1496	0.1563	0.1563	1.0000
6	1.25	0.1456	0.1547	0.2416	1.5625
7	1.50	0.1418	0.1530	0.3443	2.2500
8	1.75	0.1382	0.1515	0.4638	3.0625
9	2.00	0.1348	0.1499	0.5998	4.0000
10	2.25	0.1316	0.1485	0.7517	5.0625

Using Row 7 of the table above as an example...

Column	Calculation
C	$0.2500^2 + (-0.9163)^2 \times [\mathrm{Exp}\{-0.1250 \times 1.50\} - \mathrm{Exp}\{-2 \times 0.1250 \times 1.50\}] \times 1.5^{-1} = 0.1418$
D	$0.0300 + \sqrt{0.1418} \times 0.3267 = 0.1530$
E	$0.1530 \times 1.50^2 = 0.3443$
F	$1.50^2 = 2.2500$

The Answer to Question 2:

Using Equation (16) above and setting T = 20 years the ordinary least squares estimate of the regression parameter β is...

$$\beta = -\sum \text{Column E} \Bigg/ \sum \text{Column F} = -\frac{1,230.96}{10,467.50} = -0.1176 \tag{20}$$

Using Equations (19) and (20) above the enterprise value of our hypothetical company is...

$$\begin{aligned} V_0 &= 1000 \times \left[-0.40 \times \left(0.0175 - 0.1176 + \frac{1}{2} \times 0.25^2\right)^{-1} - 0.60 \times \left(0.0175 - 0.1250 - 0.1176 + \frac{1}{2} \times 0.25^2\right)^{-1}\right] \\ &= \$8,919,000 \end{aligned} \tag{21}$$

Summary

The value of our hypothetical company is...

Valuation Method	Company Value	Reference
Numerical integration	$\$8,770,000$	Equation (7)
Ordinary least squares	$\$8,919,000$	Equation (21)

References

[1] Gary Schurman, *Modeling And Pricing Event Risk - Part I,* August, 2017.

[2] Gary Schurman, *Modeling And Pricing Event Risk - Part II,* August, 2017.

Appendix

A. The solution to the first integral in Equation (18) above is...

$$\int_0^\infty \mathrm{Exp}\left\{\left(\mu+\beta+\frac{1}{2}\,\sigma^2\right)t\right\}\delta t=\left(\mu+\beta+\frac{1}{2}\,\sigma^2\right)^{-1}\mathrm{Exp}\left\{\left(\mu+\beta+\frac{1}{2}\,\sigma^2\right)t\right\}\Bigg[_0^\infty$$

$$=\left(\mu+\beta+\frac{1}{2}\,\sigma^2\right)^{-1}\left(0-1\right)$$

$$=-\left(\mu+\beta+\frac{1}{2}\,\sigma^2\right)^{-1}\ \text{...when...}\ \mu+\beta+\frac{1}{2}\,\sigma^2\ \text{is negative} \tag{22}$$

B. The solution to the second integral in Equation (18) above is...

$$\int_0^\infty \mathrm{Exp}\left\{\left(\mu-\lambda+\beta+\frac{1}{2}\,\sigma^2\right)t\right\}\delta t=\left(\mu-\lambda+\beta+\frac{1}{2}\,\sigma^2\right)^{-1}\mathrm{Exp}\left\{\left(\mu-\lambda+\beta+\frac{1}{2}\,\sigma^2\right)t\right\}\Bigg[_0^\infty$$

$$=\left(\mu-\lambda+\beta+\frac{1}{2}\,\sigma^2\right)^{-1}\left(0-1\right)$$

$$=-\left(\mu-\lambda+\beta+\frac{1}{2}\,\sigma^2\right)^{-1}\ \text{...when...}\ \mu-\lambda+\beta+\frac{1}{2}\,\sigma^2\ \text{is negative} \tag{23}$$